LOUGHBOROUGH UNIVERSITY

Department of Mathematical Sciences

TABLES OF FORMULAE

Loughborough University

CONTENTS

Set Algebra

$A, B, C, \ldots$ denote sets, and $a, b, c, \ldots$ their members.

The member x belongs to the set A is denoted symbolically by $x \in A$.

The **empty set** is denoted by $\emptyset$; the **universal set** is denoted by S.

A is a **subset** of B, written $A \subseteq B$ means that $x \in A$ implies $x \in B$.

Two sets A, B are equal, and we write $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

The **complement** of A is denoted by A'. This is the set $\{x \in S : \; x \notin A\}$.

The **union** of A with B is denoted by $A \cup B$. This is the set $\{x \in S : \; x \in A \text{ or } x \in B\}$

The **intersection** of A and B is denoted $A \cap B$. This is the set $\{x \in S : \; x \in A \text{ and } x \in B\}$

Laws of Combination

Inclusion $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$.

Commutative $A \cup B = B \cup A$, $A \cap B = B \cap A$.

Associative $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

Distributive $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Idempotent $A \cup A = A$, $A \cap A = A$

Complement $A \cup A' = S$, $A \cap A' = \emptyset$, $(A')' = A$.

De Morgan's Laws $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$.

Relations between elements of a set For set S and a relation $\sim$:

The relation is *reflexive* if, for all $a \in S$ $\; a \sim a$

The relation is *symmetric* if, for all $a, b \in S$ $\; a \sim b$ implies $b \sim a$.

The relation is *transitive* if, for all $a, b, c \in S$ $\; a \sim b$ and $b \sim c$ implies $a \sim c$.

An *equivalence relation* is one which is reflexive, symmetric and transitive.

Algebraic Results

Indices

$a^0 = 1, \quad a^m \times a^n = a^{m+n}, \quad \dfrac{a^m}{a^n} = a^{m-n}, \quad (a^m)^n = a^{mn} = (a^n)^m, \quad a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$

Identities

$(a+b)^2 = a^2 + 2ab + b^2$, $\quad (a-b)^2 = a^2 - 2ab + b^2$

$(a+b)(a-b) = a^2 - b^2$

$(a+b)(a^2 - ab + b^2) = a^3 + b^3$, $\quad (a-b)(a^2 + ab + b^2) = a^3 - b^3$

Completing the square: $\quad ax^2 + bx + c = a(x + \dfrac{b}{2a})^2 + c - \dfrac{b^2}{4a}$

ALGEBRAIC STRUCTURES

For a binary operation $*$ on a set S the following terms are defined:

The set is *closed* if for all $a, b \in S$ then $a * b \in S$.

An *identity* element for $*$ in S is an element $i \in S$ such that for all $a \in S$ $\quad i * a = a * i = a$.

The *inverse* of an element a is an element $a^{-1} \in S$ such that $\quad a * a^{-1} = a^{-1} * a = i$.

The operation $*$ is *associative* if, for all $a, b, c \in S$ then $(a * b) * c = a * (b * c)$.

The operation $*$ is *commutative* if, for all $a, b \in S$ then $a * b = b * a$.

For **two** operations $*$, $\circ$:

the operation $*$ is *distributive* over the operation $\circ$ if, for all $a, b, c \in S$,

$$a * (b \circ c) = (a * b) \circ (a * c) \quad \text{and} \quad (b \circ c) * a = (b * a) \circ (c * a).$$

Group $(G, *)$ There is a single associative binary operation $*$ for which G is closed under the operation $*$. An identity element exists and every element has an inverse.

An *isomorphism* between two groups $(G, *)$ and $(H, \circ)$ is a one–one mapping such that if $a \leftrightarrow \alpha$ and $b \leftrightarrow \beta$ (where $a, b \in G$; $\alpha, \beta \in H$) then $a * b \leftrightarrow \alpha \circ \beta$.

A *homomorphism* between two groups $(G, *)$ and $(H, \circ)$ is a mapping such that if $a \rightarrow \alpha$ and $b \rightarrow \beta$ (where $a, b \in G$; $\alpha, \beta \in H$) then $a * b \rightarrow \alpha \circ \beta$.

Abelian Group This is a group in which the operation $*$ is also commutative.

Ring $(S, +, \circ)$ This is an abelian group under operation $+$ (with the identity written as 0 and the inverse of a as $-a$). Under the operation $\circ$ the set S is closed and associative. The operation $\circ$ is distributive over the operation $+$.

Integral Domain $(S, +, \circ)$ This is a ring under the operation $\circ$ which is commutative. The identity exists (denoted by 1). Also the cancellation property holds:

$$\text{if} \quad a \neq 0 \quad \text{then} \quad a \circ b = a \circ c \quad \text{implies} \quad b = c$$

Field $(F, +, \circ)$ This is a set which is an abelian group under the operation $+$ (with the identity written as 0 and the inverse of a is written as $-a$). Also, under the operation $\circ$ the elements (other than 0) form an abelian group (with the identity written as 1 and the inverse of a written as a^{-1}). Also the operation $\circ$ is distributive over $+$.

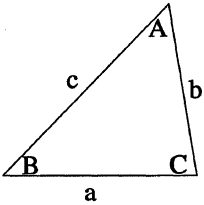
Vector (or Linear) Space $V = \{\mathbf{a}, \mathbf{b}, \mathbf{c} \ldots\}$ over a field $F = (\{\lambda, \mu, \nu \ldots\}, +, \circ)$.

This is an abelian group under the operation $+$ and which also satisfies, for all $\lambda, \mu \in F$ and for all $\mathbf{a}, \mathbf{b} \in V$

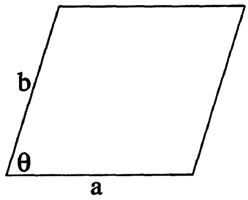
$$\lambda\mathbf{a} \in V \qquad \lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b} \qquad (\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a} \qquad (\lambda \circ \mu)\mathbf{a} = \lambda(\mu\mathbf{a}) \qquad 1\mathbf{a} = \mathbf{a}$$

Linear Dependence Vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent if scalars λ, μ, ν not all zero exist such that $\lambda\mathbf{u} + \mu\mathbf{v} + \nu\mathbf{w} = \mathbf{0}$ (and similarly for more or fewer than three vectors).

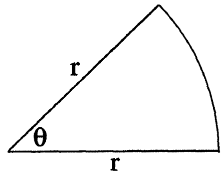
Areas and Volumes



triangle



parallelogram



sector of a circle

Area of triangle is $\frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$ or

area $= \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$.

Area of parallelogram is $ab\sin\theta$.

Area of sector of circle is $\frac{1}{2}r^2\theta$ where θ is in radians.

Curved Surface Areas

Right circular cylinder $S =$ (circumference)×(height).

Right circular cone $S = \pi\times$(radius)×(slant height).

Sphere $S = 4\pi(\text{radius})^2$.

Volumes

Right pyramid $V = \frac{1}{3}$(base area×height).

Right circular cone $V = \frac{1}{3}$(base area×height).

Right circular cylinder $V =$ base area×height.

Sphere $V = \frac{4}{3}\pi(\text{radius})^3$.

PROPERTIES OF LOGARITHMS

For any positive base $a \neq 1$, the expression $\log_a x = y$ means $a^y = x$.

$\log_e x$ is also written $\ln x$ $\log_a 1 = 0$ $\log_a a = 1$

$\log_a(xy) = \log_a x + \log_a y$, $\log_a(x/y) = \log_a x - \log_a y$

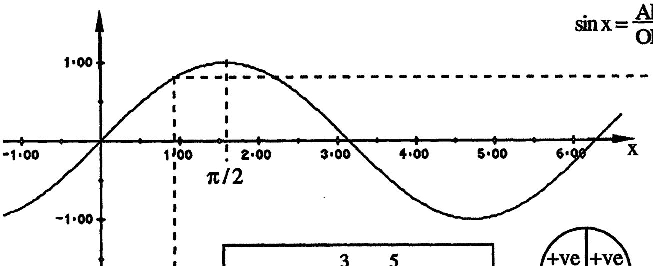
$\log_a x^n = n\log_a x$ $\log_b x = \log_a x / \log_a b = \log_a x . \log_b a$

$a^{\log_a x} = x$ $\log_a a^x = x$.

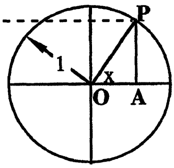
TRIGONOMETRIC FUNCTIONS

AERO4

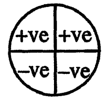
$y = \sin x$



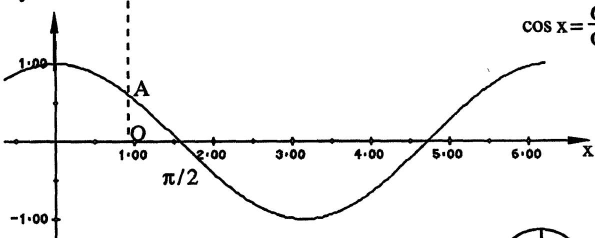
$$\sin x = \frac{AP}{OP}$$



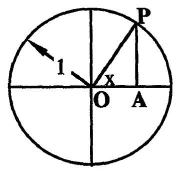
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots$$



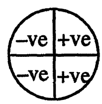
$y = \cos x$

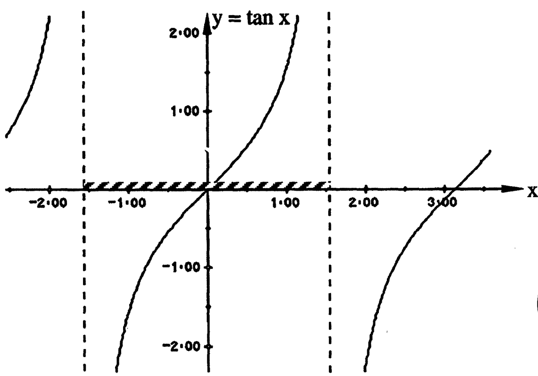


$$\cos x = \frac{OA}{OP}$$

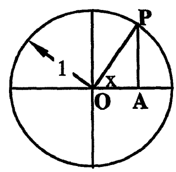


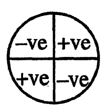
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$$



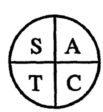


$$\tan x = \frac{AP}{OA}$$





$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots\ldots\quad -\pi/2 < x < \pi/2$$



SOME TRIGONOMETRIC IDENTITIES

$$\sin(90^o - a) \equiv \cos a \equiv (\sin 90^o + a)\ , \qquad \cos(90^o - a) \equiv \sin a \equiv -\cos(90^o + a).$$

$$\sin(a+b) \equiv \sin a \cos b + \cos a \sin b\ , \qquad \cos(a+b) \equiv \cos a \cos b - \sin a \sin b.$$

$$\tan a \equiv \frac{\sin a}{\cos a}\ , \qquad \sec\, a \equiv \frac{1}{\cos a} \qquad \text{cosec}\ a \equiv \frac{1}{\sin a}\ , \qquad \cot\ a \equiv \frac{\cos a}{\sin a}.$$

$$\cos^2 a + \sin^2 a \equiv 1\ , \qquad 1 + \tan^2 a \equiv \sec^2 a\ , \qquad 1 + \cot^2 a \equiv \text{cosec}^2 a.$$

$$\sin 2a \equiv 2 \sin a \cos a\ , \qquad \cos 2a \equiv \cos^2 a - \sin^2 a \equiv 2\cos^2 a - 1 \equiv 1 - 2\sin^2 a.$$

$$\tan(a+b) \equiv \frac{\tan a + \tan b}{1 - \tan a \tan b}\ , \qquad \tan 2a \equiv \frac{2\tan a}{1 - \tan^2 a}.$$

$$\sin a + \sin b \equiv 2 \sin\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)\ , \qquad 2 \sin a \cos b \equiv \sin(a+b) + \sin(a-b).$$

$$\cos a + \cos b \equiv 2 \cos\left(\frac{a+b}{2}\right) \cos\left(\frac{a-b}{2}\right)\ , \qquad 2 \cos a \cos b \equiv \cos(a-b) + \cos(a+b).$$

$$\sin a - \sin b \equiv 2 \cos\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)\ , \qquad 2 \cos a \sin b \equiv \sin(a+b) - \sin(a-b).$$

$$\cos a - \cos b \equiv -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)\ , \qquad 2 \sin a \sin b \equiv \cos(a-b) - \cos(a+b).$$

HYPERBOLIC FUNCTIONS

$$\sinh x \equiv \frac{1}{2}(e^x - e^{-x}) \ ; \quad \cosh x \equiv \frac{1}{2}(e^x + e^{-x}).$$

$$\text{sech}\ x \equiv \frac{1}{\cosh x} \qquad \text{cosech}\ x \equiv \frac{1}{\sinh x}$$

$$\tanh x \equiv \frac{\sinh x}{\cosh x} \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} \equiv \frac{e^{2x} - 1}{e^{2x} + 1} \equiv \frac{1 - e^{-2x}}{1 + e^{-2x}}. \qquad \coth x \equiv \frac{1}{\tanh x}$$

$$\cosh^2 x - \sinh^2 x \equiv 1\ , \qquad 1 - \tanh^2 x \equiv \text{sech}^2 x\ , \qquad \coth^2 x - 1 \equiv \text{cosech}^2 x.$$

$$\sinh^{-1}\left(\frac{x}{a}\right) \equiv \log_e\left(\frac{x + \sqrt{x^2 + a^2}}{a}\right)\ , \qquad \cosh^{-1}\left(\frac{x}{a}\right) \equiv \log_e\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right) \qquad x \geq a.$$

$$\tanh^{-1}\left(\frac{x}{a}\right) \equiv \frac{1}{2} \log_e\left(\frac{a+x}{a-x}\right) \qquad |x| < a.$$

Osborne's Rule

To obtain hyperbolic identities from corresponding trigonometric identities, replace sin by sinh and cos by cosh, except (sin × sin) which is replaced by −(sinh × sinh).

CO-ORDINATE GEOMETRY

The distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

The area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) in the (x, y) plane is

$$\frac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\} = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

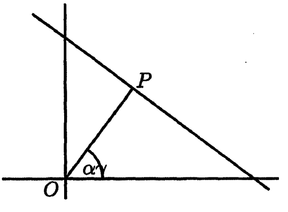
Equations of a Line

General formulation $\quad ax + by + c = 0.$

Slope m and intercept h $\quad y = mx + h.$

Intercepts g and h $\quad \dfrac{x}{g} + \dfrac{y}{h} = 1.$

Perpendicular distance from origin and normal angle: $x\cos\alpha + y\sin\alpha = |OP|$.



Slope m and point (x_1, y_1) $\quad y - y_1 = m(x - x_1).$

Two points (x_1, y_1), (x_2, y_2) $\quad \dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}.$

Angle θ between two lines $\quad y = m_1x + c_1,\ y = m_2x + c_2$;

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1\, m_2}$$

Parallel lines $\quad m_1 = m_2$. Perpendicular lines $\quad m_1m_2 = -1$

Perpendicular distance from (x_1, y_1) to $ax + by + c = 0$ is $\quad d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

Circle

$(x-x_1)^2+(y-y_1)^2=a^2$, centre (x_1,y_1) and radius a.

General form $x^2+y^2+2gx+2fy+c=0$, centre $(-g,-f)$, radius $\sqrt{(g^2+f^2-c)}$.

Tangent at (x_1,y_1): $\quad xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$

Parabola

$y^2=4ax$ or in parametric form: $x=at^2,\ y=2at.$

Tangent at (x_1,y_1) has equation $\quad yy_1=2a(x+x_1)\quad$ or at $t_1\quad t_1y=x+at_1^2$

Ellipse (in standard position)

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ or in parametric form: $\quad x=a\cos t,\ \ y=b\sin t\quad (a>b).$

Tangent at (x_1,y_1): $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1$ or at $t_1\quad \dfrac{x\cos t_1}{a}+\dfrac{y\sin t_1}{b}=1.$

Hyperbola (in standard position)

$\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1\quad$ or in parametric form: $\quad x=a\cosh t,\ y=b\sinh t$ (right hand branch).

Tangent at (x_1,y_1): $\dfrac{xx_1}{a^2}-\dfrac{yy_1}{b^2}=1$ or at $t_1\quad \dfrac{x\cosh t_1}{a}-\dfrac{y\sinh t_1}{b}=1.$

Rectangular hyperbola (with asymptotes as coordinate axes)

$xy=c^2\quad$ or in parametric form: $\quad x=ct,\ \ y=c/t.$

Tangent at (x_1,y_1): $xy_1+x_1y=2c^2$ or at $t_1\quad x+t_1^2y=2ct_1.$

Equation of a plane

$ax+by+cz+d=0.$

perpendicular distance from (x_1,y_1,z_1) to this plane is $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}.$

General second degree equation $\quad ax^2+2hxy+by^2+2gx+2fy+c=0$ can be written in the matrix form $X^TAX=0$ where

$$A=\begin{bmatrix} a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\qquad\text{and}\qquad X=\begin{bmatrix} x\\ y\\ 1\end{bmatrix}$$

Conditions for particular conics (a,h,b not all zero)

Parabola $\quad ab-h^2=0$

Ellipse $\quad ab-h^2>0.$

Hyperbola $\quad ab-h^2<0.$

Line-pair $\quad \det A=0.$

Circle $\quad a=b$ and $h=0.$

Rectangular hyperbola $\quad a+b=0.$

COMPLEX NUMBERS

$$j = \sqrt{-1} \qquad j^2 = -1 \qquad \text{alternatively the symbol } i \text{ may be used}$$

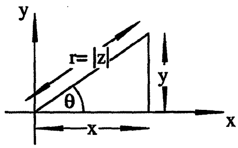
Forms of a complex number:

$z = x + jy$: *Cartesian form.*

$z = (x, y)$: *Co-ordinate form.*

$z = r(\cos\theta + j\sin\theta)$: *Polar form.*

$z = re^{j\theta}$: *Exponential form.*



The **modulus** of z is denoted by $|z|$ where $|z| = r = \sqrt{x^2 + y^2}$.

The **argument** of z is denoted by $\arg(z)$ and is the angle θ.

The **complex conjugate** of z is denoted by $\bar{z}$ and $\bar{z} = x - jy = r(\cos\theta - j\sin\theta) = re^{-j\theta}$

$z\bar{z} = |z|^2 = x^2 + y^2 = r^2$.

$z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$.

$z_1 z_2 = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1) = r_1 r_2(\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2))$.

$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2))$.

$|z_1 z_2| = |z_1||z_2| \qquad \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$; $\qquad$ triangle inequality $\quad |z_1 + z_2| \le |z_1| + |z_2|$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$; $\qquad \arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

De Moivre's Theorem

For positive integer n

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$$

For rational p/q **one** value of $(\cos\theta + j\sin\theta)^{p/q}$ is $\quad \cos\left(\dfrac{p}{q}\theta\right) + j\sin\left(\dfrac{p}{q}\theta\right)$

Relations between exponential, trigonometric and hyperbolic functions:

$$e^{j\theta} \equiv \cos\theta + j\sin\theta \qquad \cos\theta \equiv \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) \qquad \sin\theta \equiv \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$\cos j\theta \equiv \cosh\theta \qquad \cosh j\theta \equiv \cos\theta \qquad \sin j\theta \equiv j\sinh\theta \qquad \sinh j\theta \equiv j\sin\theta$$

DIFFERENTIATION AND DERIVATIVES

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx} \qquad \frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} \qquad \frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

function of a function $\quad \frac{d}{dx}(f[g(x)]) = \frac{df}{dg}\frac{dg}{dx} = f'(g(x))g'(x)$

Function	**Derivative**
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
e^{ax}	$a\,e^{ax}$
$\log_e \lvert x \rvert$	$\frac{1}{x}$
$\sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\cos^{-1}\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2 - x^2}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{(a^2 + x^2)}$
a^x	$a^x \log_e a$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\coth x$	$-\operatorname{cosech}^2 x$

INTEGRATION AND INTEGRALS

Note: $\int f(x)dx = F(x) + c$ where c is an arbitrary constant.

Function $f(x)$	Indefinite Integral $F(x)$	Function $f(x)$	Indefinite Integral $F(x)$
x^n	$\frac{x^{n+1}}{n+1},\ n \neq -1$	$\sec^2 x$	$\tan x$
$\frac{1}{x}$	$\log_e \lvert x\rvert \quad x \neq 0$	$\operatorname{cosec}^2 x$	$-\cot x$
e^{ax}	$\frac{1}{a}e^{ax}$	$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\sin x$	$-\cos x$	$\frac{1}{x^2-a^2}$	$-\frac{1}{a}\coth^{-1}\frac{x}{a} \quad \lvert x\rvert > a$
$\cos x$	$\sin x$	$\frac{1}{a^2-x^2}$	$\frac{1}{a}\tanh^{-1}\frac{x}{a} \quad \lvert x\rvert < a$
$\tan x$	$\log_e \lvert\sec x\rvert$	$\frac{1}{\sqrt{a^2+x^2}}$	$\sinh^{-1}\frac{x}{a}$
$\sinh x$	$\cosh x$	$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\frac{x}{a} \quad \lvert x\rvert > a$
$\cosh x$	$\sinh x$	$\tanh x$	$\log_e(\cosh x)$
$\operatorname{cosec} x$	$\log_e \lvert\operatorname{cosec} x - \cot x\rvert$	$\operatorname{sech}^2 x$	$\tanh x$
$\sec x$	$\log_e \lvert\sec x + \tan x\rvert$	$\operatorname{cosech}^2 x$	$-\coth x$
$\cot x$	$\log_e \lvert\sin x\rvert$	$f(ax+b)$	$\frac{1}{a}F(ax+b) \quad a \neq 0$

Integration by Parts: there are two basic forms:

$$\int u\ dv = uv - \int v\ du \qquad \text{or} \qquad \int fg\ dx = f\int g\ dx - \int \frac{df}{dx}\left\{\int g\ dx\right\} dx$$

The t substitution: used for integrating rational functions of $\sin x$ and $\cos x$.

If $t = \tan\frac{x}{2}$ then $\quad \sin x = \frac{2t}{(1+t^2)}, \quad \cos x = \frac{(1-t^2)}{(1+t^2)}, \quad dx = \frac{2dt}{1+t^2}.$

Wallis's Formulae

(a) $I_n = \int_0^{\pi/2} \sin^n\theta\ d\theta = \int_0^{\pi/2} \cos^n\theta\ d\theta = \frac{n-1}{n}I_{n-2} \qquad I_1 = 1 \qquad I_0 = \frac{\pi}{2}$

(b) $I_{m,n} = \int_0^{\pi/2} \sin^m\theta\cos^n\theta\ d\theta = \frac{m-1}{m+n}I_{m-2,n} = \frac{n-1}{m+n}I_{m,n-2} \qquad m,n \geq 0$

with $I_{0,0} = \frac{\pi}{2} \quad I_{0,1} = I_{1,0} = 1 \quad I_{1,1} = \frac{1}{2}$

SERIES

Arithmetic progression

$$S_n = a + (a+d) + (a+2d) + \ldots + (a + [n-1]d) = na + \tfrac{1}{2}n(n-1)d$$

Geometric Series

$$S_n = a + ar + ar^2 + \ldots ar^{n-1} = a(1-r^n)/(1-r) \qquad r \neq 1$$

If $|r| < 1$, $S_\infty = a/(1-r)$

Powers of natural numbers

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n+1)^2$$

Factorials

$$n! = n(n-1)(n-2)\ldots(3)(2)1 \qquad 1! = 1 \qquad 0! = 1$$

Binomial Coefficients

$${}^nC_r = \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} = \frac{n!}{(n-r)!r!} \qquad {}^nC_0 = 1 \qquad {}^nC_n = 1 \qquad {}^nC_r = {}^nC_{n-r}$$

For large values of n, Stirling's approximation is:

$$n! \approx \sqrt{2\pi} n^{n+\frac{1}{2}} e^{\left(-n+\frac{1}{12n}\right)}$$

Binomial Series

If n is a positive integer

$$(a+b)^n = a^n + nba^{n-1} + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r} b^r a^{n-r} + \ldots + b^n$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{1.2} + \ldots + \frac{n(n-1)\ldots(n-r+1)x^r}{1.2\ldots r} + \ldots + x^n$$

or,

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$$

Maclaurin Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^r}{r!}f^{(r)}(0) + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ (valid for all } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad \text{(all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad \text{(all } x)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots \qquad (|x| < \tfrac{\pi}{2})$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \qquad \text{(all } x)$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \qquad \text{(all } x)$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \ldots \qquad \text{(all } x)$$

$$\log_e(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \qquad (-1 < x \leq 1)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \qquad (-1 \leq x \leq 1)$$

$$(1 + x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \ldots + \frac{p(p-1)(p-2)\ldots(p-r+1)}{r!}x^r + \ldots$$

(for $|x| < 1$ if p is **not** a positive integer)

Taylor Series

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots + \frac{(x-a)^r}{r!}f^{(r)}(a) + \ldots$$

or (with $x = a + h$)

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^r}{r!}f^{(r)}(a) + \ldots$$

VECTORS

Let $\mathbf{a} = (a_1, a_2, a_3) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = (b_1, b_2, b_3) = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ (in Cartesians).

The length or **magnitude** of a vector $\mathbf{a}$ is denoted by $|\mathbf{a}|$.

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Vector Algebra

The **dot** or **scalar product** of $\mathbf{a}$ and $\mathbf{b}$ is written $\mathbf{a} \cdot \mathbf{b}$ and

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 \quad \text{or} \quad \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. The scalar product is a **scalar.**

The **cross** or **vector product** of $\mathbf{a}$ and $\mathbf{b}$ is written $\mathbf{a} \times \mathbf{b}$ (or $\mathbf{a} \wedge \mathbf{b}$) and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

alternatively $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and is such that $\mathbf{a}, \mathbf{b}$ and $\hat{\mathbf{n}}$, when taken in this order, form a right-handed system. The vector product is a **vector.**

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \qquad (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

Vector Calculus (in Cartesian coordinates)

$$\text{grad} \equiv \nabla \qquad \text{div} \equiv \nabla \cdot \qquad \text{curl} \equiv \nabla \times \qquad \text{Laplacian} \equiv \nabla^2$$

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \qquad \nabla^2 \equiv \text{div(grad)} \equiv \nabla \cdot \nabla \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\text{grad}\,\phi \equiv \nabla\phi \equiv \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}: \qquad \text{grad}\,\phi \text{ is a } \textbf{vector}$$

$$\text{div}\,\mathbf{F} \equiv \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \text{div}\,\mathbf{F} \text{ is a } \textbf{scalar}$$

$$\text{curl}\,\mathbf{F} \equiv \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad \text{curl}\,\mathbf{F} \text{ is a } \textbf{vector}$$

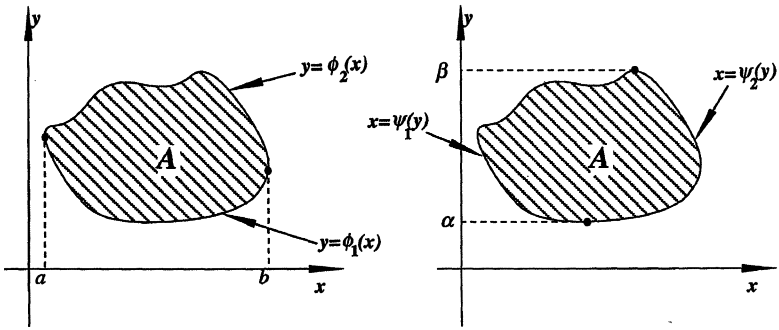
Relations in general coordinates

$$\begin{aligned}
\operatorname{grad}(\phi\psi) &= \phi \operatorname{grad} \psi + \psi \operatorname{grad} \phi \equiv \phi\nabla\psi + \psi\nabla\phi \\
\operatorname{div}(\phi\mathbf{F}) &= \phi \operatorname{div} \mathbf{F} + \mathbf{F}\cdot \operatorname{grad} \phi \equiv \phi\nabla\cdot\mathbf{F} + \mathbf{F}\cdot\nabla\phi \\
\operatorname{curl}(\phi\mathbf{F}) &= \phi \operatorname{curl} \mathbf{F} + \operatorname{grad} \phi \times \mathbf{F} \equiv \phi\nabla\times\mathbf{F} + \nabla\phi\times\mathbf{F}
\end{aligned}$$

$$\operatorname{curl} \operatorname{grad} \phi \equiv \mathbf{0} \qquad \operatorname{div} \operatorname{curl} \mathbf{F} \equiv 0$$

$$\begin{aligned}
\operatorname{curl} \operatorname{curl} \mathbf{F} &= \operatorname{grad} \operatorname{div} \mathbf{F} - \boldsymbol{\nabla}^2\mathbf{F} \equiv \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F} \\
\operatorname{grad}(\mathbf{F}\cdot\mathbf{G}) &= (\mathbf{G}\cdot\operatorname{grad})\mathbf{F} + (\mathbf{F}\cdot\operatorname{grad})\mathbf{G} + \mathbf{G}\times\operatorname{curl} \mathbf{F} + \mathbf{F}\times\operatorname{curl} \mathbf{G} \\
\operatorname{div}(\mathbf{F}\times\mathbf{G}) &= \mathbf{G}\cdot\operatorname{curl} \mathbf{F} - \mathbf{F}\cdot\operatorname{curl} \mathbf{G} \\
\operatorname{curl}(\mathbf{F}\times\mathbf{G}) &= (\mathbf{G}\cdot\operatorname{grad})\mathbf{F} - (\mathbf{F}\cdot\operatorname{grad})\mathbf{G} + \mathbf{F} \operatorname{div} \mathbf{G} - \mathbf{G} \operatorname{div} \mathbf{F}
\end{aligned}$$

Double Integrals



$$\iint_A f(x,y)dxdy = \int_a^b \left\{ \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right\} dx = \int_\alpha^\beta \left\{ \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right\} dy$$

Green's Theorem in the Plane

$$\oint_C (Pdx + Qdy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

Stokes' Theorem

$$\oint_C (\mathbf{A}\cdot d\mathbf{r}) = \iint_S \operatorname{curl}\mathbf{A}\cdot d\mathbf{S}$$

Divergence Theorem

$$\oiint_S (\mathbf{A}\cdot d\mathbf{S}) = \iiint_V \operatorname{div}\mathbf{A}\; dV$$

Determinants and Matrices

Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - hf) - b(di - gf) + c(dh - ge)$$

Matrices

The **transpose** of a $p \times q$ matrix M is denoted by M^T and is a $q \times p$ matrix found by interchanging the rows and columns of M.

If A is a *square* matrix then A is said to be **symmetric** if $A = A^T$.

The **trace** of a square matrix A is equal to the sum of the terms on the leading diagonal.

The **adjoint** matrix of a square matrix A is denoted by $\text{adj}(A)$ and is the **transposed matrix of cofactors.**

The **inverse** of A exists if $\det A \neq 0$ and then $A^{-1} = \dfrac{1}{\det(A)}\text{adj}(A)$.

$$\text{if} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

A is said to be **orthogonal** if $A^T = A^{-1}$.

Two $n \times 1$ column vectors X, Y are said to be **orthogonal** if $X^T Y = 0$

Eigenvalues and Eigenvectors

An **eigenvector** of A is a non-zero column vector X such that $AX = \lambda X$ where λ (a scalar) is the corresponding **eigenvalue.** The eigenvalues satisfy the *characteristic equation*

$$\det(A - \lambda I) = 0$$

The sum of the eigenvalues of A is equal to the trace of A.

The product of the eigenvalues of A is equal to the determinant of A.

An $n \times n$ symmetric matrix A with real elements has only **real** eigenvalues and n independent eigenvectors of A can be found. The eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal.

The **modal matrix** corresponding to the $n \times n$ square matrix A is an $n \times n$ square matrix P whose columns are the eigenvectors of A. If n independent eigenvectors are used to form P then $P^{-1}AP$ is a diagonal matrix in which the diagonal entries are the eigenvalues of A taken in the same order that the eigenvectors were taken to form P.

Differential and Difference Equations

First-order linear Differential equations

The integrating factor for $\dfrac{dy}{dx} + p(x)y = q(x)$ is $r(x) = \exp\left\{\int_c^x p(t)dt\right\}$ which then reduces the differential equation to the form $\dfrac{d}{dx}\{r(x)y\} = r(x)q(x)$.

Second Order Linear Equations

DifferentialEquations	**DifferenceEquations**

The *auxiliary* equation for

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0 \qquad\qquad u_{n+2} + au_{n+1} + bu_n = 0$$

where a, b are constants, is

$$m^2 + am + b = 0.$$

If the auxiliary equation has distinct roots α and β, the solution, called the *complementary function*, is

$$y = Ae^{\alpha x} + Be^{\beta x} \qquad\qquad u_n = A\alpha^n + B\beta^n$$

where A, B are constants. If the roots are complex then use the expansion $e^{(c+jd)} = e^c(\cos d + j\sin d)$. If the auxiliary equation has only one distinct root α then the complementary function is

$$y = (A + Bx)e^{\alpha x} \qquad\qquad u_n = (A + Bn)\alpha^n$$

The general solution of

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = f(x) \qquad\qquad u_{n+2} + au_{n+1} + bu_n = f(n)$$

is the sum of any particular solution and the complementary function.

For the differential equation and for some continuous function $f(x)$ then:

If the *initial values* of both y and $\dfrac{dy}{dx}$ are given for some x_0, there is sure to be a unique solution in some domain including x_0.

If the *boundary values* of either y and $\dfrac{dy}{dx}$ are given at two values of x, there may be no solution, or one solution, or an infinity of solutions.

LAPLACE TRANSFORMS

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt \equiv F(s). \quad \text{In all cases, } f(t) = 0 \text{ for } t < 0.$$

In this table $u(t)$ is the unit step (Heaviside) function:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Function f(t)	**Transform F(s)**
t^n	$\dfrac{n!}{s^{n+1}}$
e^{-at}	$\dfrac{1}{(s+a)}$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$t^n f(t)$	$(-1)^n \dfrac{d^n F(s)}{ds^n}$
$\int_0^t f(x)\, dx$	$\dfrac{1}{s} F(s)$
$e^{-at} f(t)$	$F(s+a)$
$\sin \omega t$	$\dfrac{\omega}{(s^2+\omega^2)}$
$\cos \omega t$	$\dfrac{s}{(s^2+\omega^2)}$
$\sinh \omega t$	$\dfrac{\omega}{(s^2-\omega^2)}$
$\cosh \omega t$	$\dfrac{s}{(s^2-\omega^2)}$
delta function: $\delta(t-T)$	e^{-sT}
$f(t-T)\, u(t-T)$	$e^{-sT} F(s)$

Also, the **convolution** of $f(t)$ and $g(t)$ is denoted by $(f * g)(t)$ and is defined by:

$$(f * g)(t) = \int_0^t f(x) g(t-x) dx$$

The convolution theorem states

$$L[(f * g)(t)] = F(s)G(s)$$

FOURIER SERIES

For a periodic signal $f(t)$ with **period** T and **angular frequency** $\omega \equiv \dfrac{2\pi}{T}$

$$\begin{aligned} f(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos n\omega t + b_n \sin n\omega t\} \\ &= \frac{1}{2}a_0 + (a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots) + (b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots) \end{aligned}$$

where

$$a_0 = \frac{2}{T}\int_d^{d+T} f(t)dt \qquad a_n = \frac{2}{T}\int_d^{d+T} f(t)\cos n\omega t\; dt \qquad b_n = \frac{2}{T}\int_d^{d+T} f(t)\sin n\omega t\; dt$$

in which d can be chosen arbitrarily.

Complex Form

Period T $\quad f(t) = \sum_{n=-\infty}^{\infty} c_n e^{(2jn\pi t/T)}$ where $\quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-(2jn\pi t/T)}dt$

FOURIER TRANSFORMS

$F(\omega)$ is the Fourier transform of $f(t)$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega \qquad (1) \qquad\qquad F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad (2)$$

Table of transforms $\;u(t)$ is the unit step function.

Function $f(t)$	**Transform** $F(\omega)$
$e^{-\alpha t}u(t)$	$\dfrac{1}{\alpha + j\omega}$
$e^{-\alpha\|t\|}$	$\dfrac{2\alpha}{\alpha^2+\omega^2}$
$P_a(t) = \begin{cases} 1 & \|t\| < a \\ 0 & \|t\| > a \end{cases}$	$\dfrac{2}{\omega}\sin \omega a$
$F(t)$	$2\pi f(-\omega)$
$f(t-t_0)$	$F(\omega)e^{-j\omega t_0}$
convolution: $(f * g)(t)$	$F(\omega)G(\omega)$
delta function: $\delta(t-t_0)$	$e^{-j\omega t_0}$

Fourier cosine transform

If f is an even function, then: $f(-s) = f(s)$ $F(-\omega) = F(\omega)$ and (1) and (2) become:

$$f(t) = \frac{2}{\pi}\int_0^{\infty} F_c(\omega)\cos\omega t\, d\omega \qquad (3) \qquad\qquad F_c(\omega) = \int_0^{\infty} f(s)\cos\omega s\, ds \qquad (4)$$

$F_c(\omega)$ is the *Fourier cosine transform* of $f(t)$.

Fourier sine transform

If f is an odd function, then: $f(-s) = -f(s)$ $F(-\omega) = -F(\omega)$ and (1) and (2) become

$$f(t) = \frac{2}{\pi}\int_0^{\infty} F_s(\omega)\sin\omega t\, d\omega \qquad (5) \qquad\qquad F_s(\omega) = \int_0^{\infty} f(s)\sin\omega s\, ds \qquad (6)$$

$F_s(\omega)$ is the *Fourier sine transform* of $f(t)$.

Parseval's Theorem:

$$\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \qquad \sum_{n=-\infty}^{\infty} f^2(n) = \frac{1}{N}\sum |F(k)|^2$$

Z-TRANSFORMS

Table of Transforms $u(n)$ is the unit step function.

Function $f(n)$	Transform $F(z)$
$f(n)$	$F(z)$
$a^n u(n)$	$\frac{z}{z-a}$
$nu(n)$	$\frac{z}{(z-1)^2}$
$f(n-1)$	$z^{-1}F(z) + f(-1)$
$f(n-2)$	$z^{-2}F(z) + z^{-1}f(-1) + f(-2)$
delta function: $\delta(n-k)$	z^{-k}
convolution: $(f * g)(n)$	$F(z)G(z)$
$(b^{n+1} - a^{n+1})/(b-a)$	$z^2/(z-a)(z-b)$
$(n+1)a^n$	$z^2/(z-a)^2$

NUMERICAL METHODS

ROOTS OF EQUATIONS

Basic Iteration

If $f(x) = 0$ is rearranged to the form: $x = F(x)$ then the scheme

$$x_{n+1} = F(x_n) \qquad n = 0, 1, 2, \ldots$$

will converge to a root if $|F'(x)| < 1$ at and near the root.

Newton-Raphson

If $f(x) = 0$ then the Newton-Raphson iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, 2, \ldots$$

FINITE DIFFERENCES

The central difference formulae for first and second order derivatives of $y = f(x)$ are

$$y'_n \approx \frac{y_{n+1} - y_{n-1}}{2h} \qquad y''_n \approx \frac{y_{n-1} - 2y_n + y_{n+1}}{h^2}$$

the forward difference approximation for a first derivative is $y'_n \approx \dfrac{y_{n+1} - y_n}{h}$.

FINITE ELEMENTS

Linear Elements

Shape functions in non-dimensional coordinates

$$L_1(\xi) = \frac{1}{2}(1 - \xi) \qquad L_2(\xi) = \frac{1}{2}(1 + \xi)$$

If $\quad M_{ij} = \displaystyle\int_{[e]} L_i(x)L_j(x)dx \quad$ then $\quad [M] = \dfrac{\ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \ell =$ element length

If $\quad K_{ij} = \displaystyle\int_{[e]} \frac{dL_i}{dx}\frac{dL_j}{dx}dx \quad$ then $\quad [K] = \dfrac{1}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Quadratic Elements

Shape functions in non-dimensional coordinates

$$Q_1(\xi) = \frac{1}{2}\xi(\xi - 1) \qquad Q_2(\xi) = 1 - \xi^2 \qquad Q_3(\xi) = \frac{1}{2}\xi(\xi + 1)$$

If $M_{ij} = \int_{[e]} Q_i(x)Q_j(x)dx$ then $[M] = \frac{\ell}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$

If $K_{ij} = \int_{[e]} \frac{dQ_i}{dx}\frac{dQ_j}{dx}dx$ then $[K] = \frac{1}{3\ell}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$

If $F_i = \int_{[e]} Q_i(x)dx$ then $\{F\} = \frac{\ell}{6}\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$

LEAST SQUARES CURVE FITTING

The *least squares straight line* through n data points (x_r, y_r) $r = 1, 2, \ldots, n$ is $y = a_0 + a_1 x$ where a_0, a_1 are found by solving the *normal equations*

$$\sum_{r=1}^{n} y_r = n a_0 + a_1 \sum_{r=1}^{n} x_r$$

$$\sum_{r=1}^{n} x_r y_r = a_0 \sum_{r=1}^{n} x_r + a_1 \sum_{r=1}^{n} x_r^2$$

The *least squares parabola* through (x_r, y_r) is $y = a_0 + a_1 x + a_2 x^2$ where a_0, a_1, a_2 are found by solving:

$$\sum_{r=1}^{n} y_r = n\, a_0 + a_1 \sum_{r=1}^{n} x_r + a_2 \sum_{r=1}^{n} x_r^2$$

$$\sum_{r=1}^{n} x_r\, y_r = a_0 \sum_{r=1}^{n} x_r + a_1 \sum_{r=1}^{n} x_r^2 + a_2 \sum_{r=1}^{n} x_r^3$$

$$\sum_{r=1}^{n} x_r^2\, y_r = a_0 \sum_{r=1}^{n} x_r^2 + a_1 \sum_{r=1}^{n} x_r^3 + a_2 \sum_{r=1}^{n} x_r^4$$

INTEGRATION

Trapezoidal rule

n strips

$$\int_a^{b=a+nh} f(x)\,dx \simeq \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f_n)$$

$$\text{error } |E_T| \leq \frac{M_T(b-a)h^2}{12}, \text{ where } M_T = \max_{a \leq x \leq b} |f''(x)|$$

Simpson's Rule

2n strips

$$\begin{aligned}\int_a^{b=a+2nh} f(x)dx &\simeq \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{2n-2} + 4f_{2n-1} + f_{2n}) \\ &= \frac{h}{3}\left(f_0 + f_{2n} + 4[f_1 + f_3 + \ldots + f_{2n-1}] + 2[f_2 + f_4 + \ldots + f_{2n-2}]\right)\end{aligned}$$

$$\text{error, } |E_S| \leq \frac{M_S(b-a)h^4}{180}, \text{ where } M_S = \max_{(a,b)} |f^{(iv)}(x)|$$

Gauss-Legendre Quadrature Formulae

The following sample points and weighting factors are used in the approximation

$$\int_{-1}^{1} f(\xi)d\xi \approx \sum_{i=1}^{n} w_i f(\xi_i)$$

order n	Sample Points ξ_i	Weights w_i
2	±0.577350	1.0
3	±0.774597	0.555556
	0.0	0.888889
4	±0.861136	0.347855
	±0.339981	0.652145
5	±0.906180	0.236927
	0.0	0.568889
	±0.538469	0.478629

ORDINARY DIFFERENTIAL EQUATIONS

First-order equations are assumed to be written $\frac{dy}{dx} = f(x, y)$ with given initial condition $y = y_0$ at $x = x_0$. We aim to find estimates $y_1, y_2, y_3, \ldots$ for the $y-$values at $x_1, x_2, x_3, \ldots$ assuming constant spacing $(x_i - x_{i-1}) = h$.

We denote $f(x_r, y_r)$ by f_r.

Euler's Method

$$y_{r+1} = y_r + hy'_r = y_r + hf(x_r, y_r)$$

Modified Euler Method *p*: predicted value, *c*: corrected value

$$y^{(p)}_{r+1} = y_r + h\ f_r \qquad f^{(p)}_{r+1} = f\left(x_{r+1},\ y^{(p)}_{r+1}\right) \qquad y^{(c)}_{r+1} = y_r + \frac{h}{2}\left(f_r + f^{(p)}_{r+1}\right)$$

Milne-Simpson Method *p*: predicted value, *c*: corrected value

$$y^{(p)}_{n+1} = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n) \qquad y^{(c)}_{n+1} = y_{n-1} + \frac{h}{3}\left(f_{n-1} + 4f_n + f^{(p)}_{n+1}\right)$$

Runge-Kutta fourth order method

$$k_1 = h\ f(x_r, y_r) \qquad k_2 = h\ f(x_r + \frac{h}{2},\ y_r + \frac{k_1}{2})$$

$$k_3 = h\ f(x_r + \frac{h}{2},\ y_r + \frac{k_2}{2}) \qquad k_4 = h\ f(x_r + h,\ y_r + k_3)$$

$$y_{r+1} = y_r + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Simultaneous first order equations

For the system

$$\frac{dy}{dt} = f(t, y, z) \qquad \frac{dz}{dt} = g(t, y, z) \qquad \text{with} \quad y(t_0) = y_0, \quad z(t_0) = z_0$$

the Runge-Kutta fourth order method is:

$$k_0 = h\ f(t_n, y_n, z_n) \qquad \ell_0 = h\ g(t_n, y_n, z_n)$$

$$k_1 = h\ f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_0, z_n + \frac{1}{2}\ell_0) \qquad \ell_1 = h\ g(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_0, z_n + \frac{1}{2}\ell_0)$$

$$k_2 = h\ f(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1, z_n + \frac{1}{2}\ell_1) \qquad \ell_2 = h\ g(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1, z_n + \frac{1}{2}\ell_1)$$

$$k_3 = h\ f(t_n + h, y_n + k_2, z_n + \ell_2) \qquad \ell_3 = h\ g(t_n + h, y_n + k_2, z_n + \ell_2)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3) \qquad z_{n+1} = z_n + \frac{1}{6}(\ell_0 + 2\ell_1 + 2\ell_2 + \ell_3)$$

PROBABILITY AND STATISTICS

Probability

A, B are events; subsets of the event space S. The event S is the certain event, and $\emptyset$ denotes the impossible event.The notation A' denotes the event 'A does not occur'.

The probability that event A occurs is denoted by $P(A)$.

The conditional probability of A occurring given that B has occurred is denoted by $P(A|B)$.

(i) $P(A') = 1 - P(A) \qquad P(S) = 1 \qquad P(\emptyset) = 0$

(ii) $P(A) = P(A \cap B) + P(A \cap B')$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(iv) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

(v) Bayes' Theorem $P(A|B) = \dfrac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A')P(A')}$

Random variables

X, Y are random variables, either discrete or continuous. If X is discrete, the probability that X should assume the value x_i is denoted by $P(X = x_i)$. If X is continuous then there exists a probability density function $f(x)$ such that the probability of X assuming a value between a and b is given by $\int_a^b f(x)\, dx$.

The **expectation** of X is denoted by $E(X)$ or by μ_X

The **variance** of X is denoted by $\text{Var}(X)$ or by σ_X^2.

(i) $\mu_X \equiv E(X) = \sum x_i P(X = x_i)$ if discrete or equals $\int_{-\infty}^{\infty} x f(x)\, dx$ if X is continuous.

(ii) $\sigma_X^2 \equiv \text{Var}(X) = \sum (x_i - \mu_X)^2 P(X = x_i)$ or $\int_{-\infty}^{\infty} (x - \mu_X)^2 f(x)\, dx$.

(iii) $\text{Var}(X) = E(X^2) - (E(X))^2$.

(iv) $E(aX + b) = aE(X) + b \qquad a, b$ constants.

(v) $\text{Var}(aX + b) = a^2 \text{Var}(X)$.

(vi) $E(X \pm Y) = E(X) \pm E(Y) \qquad \text{Var}(X \pm Y) = \text{Var}(X) + \text{Var}(Y) \pm 2\text{Cov}(X, Y)$.

where the covariance

$$\text{Cov}(X, Y) = E\{(X - \mu_X)(Y - \mu_Y)\} = E(XY) - \mu_X \mu_Y.$$

If X, Y are independent then $\text{Cov}(X, Y) = 0$ and then

$$\text{Var}(aX \pm bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y).$$

Statistics

MEAN AND VARIANCE

Populations

For a population of n quantities $x_1, x_2, \ldots x_n$

mean, $\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$ variance, $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \mu^2$

Grouped data

If the observations are arranged in s groups with frequencies $f_1, f_2, \ldots, f_s$ and mid-points x_j, then

$$n = \sum_{j=1}^{s} f_j \qquad \mu = \frac{1}{n}\sum_{j=1}^{s} f_j\, x_j \qquad \sigma^2 = \frac{1}{n}\sum_{j=1}^{s} f_j (x_j - \mu)^2 = \frac{1}{n}\sum_{j=1}^{s} f_j\, x_j^2 - \mu^2$$

Samples

Unbiased estimates of μ and σ based on a sample of n values $x_1, x_2, \ldots, x_n$ are

mean, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ variance, $s^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{(n-1)}\sum_{i=1}^{n} x_i^2 - \frac{n}{(n-1)}\bar{x}^2$

Coded variables

If $u_i = \frac{(x_i - a)}{b}$ then $\bar{u} = \frac{(\bar{x} - a)}{b}$, $\bar{x} = a + b\bar{u}$ and $\mathrm{Var}(u_i) = \frac{1}{b^2}\,(\mathrm{Var}(x_i))$.

BINOMIAL DISTRIBUTION

A Bernoulli trial has two outcomes classed as success and failure. The probability of success is p and of failure is $q = 1 - p$. The probability of r successes in a set of n identical, independent Bernoulli trials is:

$$P(X = r) = {}^{n}C_r p^r q^{(n-r)} = \frac{n!}{r!(n-r)!} p^r (1-p)^{(n-r)}$$

The expected value, $E(X) = np$ and the variance $\mathrm{Var}(X) = npq = np(1-p)$.

POISSON DISTRIBUTION

If the average number of events occurring in the stated unit of space or time is λ, then the probability of r such events occurring in that unit is given by

$$P(X = r) = \frac{\lambda^r}{r!} e^{-\lambda} \qquad E(X) = \lambda \qquad \mathrm{Var}(X) = \lambda$$

THE NORMAL PROBABILITY INTEGRAL

$z = \frac{x-\mu}{\sigma}$	0	1	2	3	4	5	6	7	8	9
0	0000	0040	0080	0120	0160	0199	0239	0279	0319	0359
.1	0398	0438	0478	0517	0577	0596	0636	0675	0714	0753
.2	0793	0832	0871	0909	0948	0987	1026	1064	1103	1141
.3	1179	1217	1255	1293	1331	1368	1406	1443	1480	1517
.4	1555	1591	1628	1664	1700	1736	1772	1808	1844	1879
.5	1915	1950	1985	2019	2054	2088	2123	2157	2190	2224
.6	2257	2291	2324	2357	2389	2422	2454	2486	2517	2549
.7	2580	2611	2642	2673	2703	2734	2764	2794	2822	2852
.8	2881	2910	2939	2967	2995	3023	3051	3078	3106	3133
.9	3159	3186	3212	3238	3264	3289	3315	3340	3365	3389
1.0	3413	3438	3461	3485	3508	3531	3554	3577	3599	3621
1.1	3643	3665	3686	3708	3729	3749	3770	3790	3810	3830
1.2	3849	3869	3888	3907	3925	3944	3962	3980	3997	4015
1.3	4032	4049	4066	4082	4099	4115	4131	4147	4162	4177
1.4	4192	4207	4222	4236	4251	4265	4279	4292	4306	4319
1.5	4332	4345	4357	4370	4382	4394	4406	4418	4429	4441
1.6	4452	4463	4474	4484	4495	4505	4515	4525	4535	4545
1.7	4554	4564	4573	4582	4591	4599	4608	4616	4625	4633
1.8	4641	4649	4656	4664	4671	4678	4686	4693	4699	4706
1.9	4713	4719	4726	4732	4738	4744	4750	4756	4761	4767
2.0	4772	4778	4783	4788	4793	4798	4803	4808	4812	4817
2.1	4821	4826	4830	4834	4838	4842	4846	4850	4854	4857
2.2	4861	4865	4868	4871	4875	4878	4881	4884	4887	4890
2.3	4893	4896	4898	4901	4904	4906	4909	4911	4913	4916
2.4	4918	4920	4922	4925	4927	4929	4931	4932	4934	4936
2.5	4938	4940	4941	4943	4946	4947	4948	4949	4951	4952
2.6	4953	4955	4956	4957	4959	4960	4961	4962	4963	4964
2.7	4965	4966	4967	4968	4969	4970	4971	4972	4973	4974
2.8	4974	4975	4976	4977	4977	4978	4979	4979	4980	4981
2.9	4981	4982	4982	4983	4984	4984	4985	4985	4986	4986
	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
	4987	4990	4993	4995	4997	4998	4998	4999	4999	4999

The Normal Probability Integral

The table shows the areas under the graph of the normal probability curve from 0 to the value of z concerned. Thus the reading 1772 corresponding to $z = 0.46$ indicates that the area under the curve from $z = 0$ to $z = 0.46$ is 0.1772.

PERCENTAGE POINTS OF THE t–DISTRIBUTION

P / ν	50	25	10	5	2.5	1	0.5	0.1
1	1.00	2.41	6.31	12.7	25.5	63.7	127.	637
2	.816	1.60	2.92	4.30	6.21	9.92	14.1	31.6
3	.765	1.42	2.35	3.18	4.18	5.84	7.45	12.9
4	.741	1.34	2.13	2.78	3.50	4.60	5.60	8.61
5	.727	1.30	2.01	2.57	3.16	4.03	4.77	6.86
6	.718	1.27	1.94	2.45	2.97	3.71	4.32	5.96
7	.711	1.25	1.89	2.36	2.84	3.50	4.03	5.40
8	.706	1.24	1.86	2.31	2.75	3.36	3.83	5.04
9	.703	1.23	1.83	2.26	2.68	3.25	3.69	4.78
10	.700	1.22	1.81	2.23	2.63	3.17	3.58	4.59
11	.698	1.21	1.80	2.20	2.59	3.11	3.50	4.44
12	.695	1.21	1.78	2.18	2.56	3.05	3.43	4.32
13	.694	1.20	1.77	2.16	2.53	3.01	3.37	4.22
14	.692	1.20	1.76	2.14	2.51	2.98	3.33	4.14
15	.691	1.20	1.75	2.13	2.49	2.95	3.29	4.07
16	.690	1.19	1.75	2.12	2.47	2.92	3.25	4.01
17	.689	1.19	1.74	2.11	2.46	2.90	3.22	3.96
18	.688	1.19	1.73	2.10	2.44	2.88	3.20	3.92
19	.688	1.19	1.73	2.09	2.43	2.86	3.17	3.88
20	.687	1.18	1.72	2.09	2.42	2.85	3.15	3.85
21	.686	1.18	1.72	2.08	2.41	2.83	3.14	3.82
22	.686	1.18	1.72	2.07	2.41	2.82	3.12	3.79
23	.685	1.18	1.71	2.07	2.40	2.81	3.10	3.77
24	.685	1.18	1.71	2.06	2.39	2.80	3.09	3.74
25	.684	1.18	1.71	2.06	2.38	2.79	3.08	3.72
26	.684	1.18	1.71	2.06	2.38	2.78	3.07	3.71
27	.684	1.18	1.70	2.05	2.37	2.77	3.06	3.69
28	.683	1.17	1.70	2.05	2.37	2.76	3.05	3.67
29	.683	1.17	1.70	2.05	2.36	2.76	3.04	3.66
30	.683	1.17	1.70	2.04	2.36	2.75	3.03	3.65
40	.681	1.17	1.68	2.02	2.33	2.70	2.97	3.55
120	.677	1.16	1.66	1.98	2.27	2.62	2.86	3.37
∞	.674	1.15	1.64	1.96	2.24	2.58	2.81	3.29

Percentage Ponits of the t – Distribution

The table gives, for the t–distribution for ν degrees of freedon, the value of $|t|$ which has a P% chance of being exceeded. Thus the entry of 2.26 corresponding to $\nu = 9$, $P = 5$, indicates a 5% chance of $|t|$ being greater than 2.26, i.e. of t being less than -2.26 or greater than $+2.26$, when the number of degrees of freedom is 9.

F. DISTRIBUTION –5%, $2\frac{1}{2}$% and 1% points

ν_1 / ν_2	1	2	3	4	5	6	7	8	10	12	24	∞
1	161 648 4052	200 800 5000	216 864 5403	225 900 5625	230 922 5764	234 937 5859	237 948 5928	239 957 5981	242 969 6056	244 977 6106	249 997 6235	254 1018 6366
2	18.5 38.5 98.5	19.0 39.0 99.0	19.2 39.2 99.2	19.3 39.2 99.2	19.4 39.3 99.3	19.4 39.3 99.3	19.4 39.4 99.4	19.4 39.4 99.4	19.4 39.4 99.4	19.4 39.4 99.4	19.5 39.5 99.5	19.5 39.5 99.5
3	10.13 17.4 34.1	9.55 16.0 30.8	9.28 15.4 29.5	9.12 15.1 28.7	9.01 14.9 28.2	8.94 14.7 27.9	8.89 14.6 27.7	8.85 14.5 27.5	8.70 14.4 27.2	8.74 14.3 27.1	8.64 14.1 26.6	8.53 13.9 26.1
4	7.71 12.22 21.2	6.94 10.65 18.0	6.59 9.98 16.7	6.39 9.60 16.0	6.26 9.36 15.5	6.16 9.20 15.2	6.09 9.07 15.0	6.04 8.98 14.8	5.96 8.84 14.5	5.91 8.75 15.5	5.77 8.51 13.9	5.63 8.26 13.5
5	6.61 10.01 16.26	5.79 8.43 13.27	5.41 7.76 12.06	5.19 7.39 11.39	5.05 7.15 10.97	4.95 6.98 10.67	4.88 6.85 10.46	4.82 6.76 10.29	4.74 6.62 10.05	4.68 6.52 9.89	4.53 6.28 9.47	4.36 6.02 9.02
6	5.99 8.81 13.74	5.14 7.26 10.92	4.75 6.60 9.78	4.53 6.23 9.15	4.39 5.99 8.75	4.28 5.82 8.47	4.21 5.70 8.26	4.15 5.60 8.10	4.06 5.45 7.87	4.00 5.37 7.72	3.84 5.12 7.31	3.67 4.85 6.88
7	5.59 8.07 12.25	4.74 6.54 9.55	4.35 5.89 8.45	4.12 5.52 7.85	3.97 5.29 7.46	3.87 5.12 7.19	3.79 4.99 6.99	3.73 4.90 6.84	3.64 4.76 6.62	3.57 4.67 6.47	3.41 4.42 6.07	3.23 4.14 5.65
8	5.32 7.57 11.26	4.46 6.06 8.65	4.07 5.42 7.59	3.84 5.05 7.01	3.69 4.82 6.63	3.58 4.65 6.37	3.50 4.53 6.18	3.44 4.43 6.03	3.35 4.30 5.81	3.28 4.20 5.67	3.12 3.95 5.28	2.93 3.67 4.86
9	5.12 7.21 10.56	4.26 5.71 8.02	3.86 5.08 6.99	3.63 4.72 6.42	3.48 4.48 6.06	3.37 4.32 5.80	3.29 4.20 5.61	3.23 4.10 5.47	3.14 3.96 5.26	3.07 3.87 5.11	2.90 3.61 4.73	2.71 3.33 4.31
10	4.96 6.94 10.04	4.10 5.46 7.56	3.71 4.83 6.55	3.48 4.47 5.99	3.33 4.24 5.64	3.22 4.07 5.39	3.14 3.95 5.20	3.07 3.85 5.06	2.98 3.72 4.85	2.91 3.62 4.71	2.74 3.37 4.33	2.54 3.08 3.91
11	4.84 6.72 9.65	3.98 5.26 7.21	3.59 4.63 6.22	3.36 4.28 5.67	3.20 4.04 5.32	3.09 3.88 5.07	3.01 3.76 4.89	2.95 3.66 4.74	2.85 3.53 4.54	2.79 3.43 4.40	2.61 3.17 4.02	2.40 2.88 3.60
12	4.75 6.55 9.33	3.89 5.10 6.93	3.49 4.47 5.95	3.26 4.12 5.41	3.11 3.89 5.05	3.00 3.73 4.82	2.91 3.61 4.64	2.85 3.51 4.50	2.75 3.37 4.30	2.69 3.28 4.16	2.51 3.02 3.78	2.30 2.72 3.36

F. DISTRIBUTION –5%, $2\frac{1}{2}$% and 1% points

ν_2 \ ν_1	1	2	3	4	5	6	7	8	10	12	24	∞
	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.60	2.53	2.35	2.13
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.15	3.05	2.79	2.49
	8.86	6.51	5.56	5.04	4.70	4.46	4.28	4.14	3.94	3.80	3.43	3.00
	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.49	2.42	2.24	2.01
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	2.99	2.89	2.63	2.32
	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.69	3.55	3.18	2.75
	4.41	3.55	3.10	2.93	2.77	2.66	2.58	2.51	2.41	2.34	2.15	1.92
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.87	2.77	2.50	2.19
	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.51	3.37	3.00	2.57
	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.35	2.28	2.08	1.84
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.77	2.68	2.41	2.02
	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.37	3.23	2.86	2.49
	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.25	2.18	1.98	1.73
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.64	2.54	2.27	1.94
	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.17	3.03	2.66	2.21
	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.19	2.12	1.91	1.65
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.55	2.45	2.17	1.83
	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.03	2.90	2.52	2.06
	4.15	3.29	2.90	2.67	2.51	2.40	2.31	2.24	2.14	2.07	1.86	1.59
32	5.53	4.15	3.56	3.22	3.00	2.84	2.72	2.62	2.48	2.38	2.10	1.75
	7.50	5.34	4.46	3.97	3.65	3.43	3.26	3.13	2.93	2.80	2.42	1.96
	4.11	3.26	2.87	2.63	2.48	2.36	2.28	2.21	2.11	2.03	1.82	1.55
36	5.47	4.09	2.51	3.17	2.94	2.79	2.66	2.57	2.43	2.33	2.05	1.69
	7.40	5.25	4.38	3.89	3.58	3.35	3.18	3.05	2.86	2.72	2.35	1.87
	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.08	2.00	1.79	1.51
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.39	2.29	2.01	1.64
	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.90	2.66	2.29	1.80
	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	1.99	1.92	1.70	1.39
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.27	2.17	1.88	1.48
	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.63	2.50	2.12	1.60
	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.01	1.83	1.61	1.25
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.16	2.05	1.76	1.31
	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.47	2.34	1.95	1.38
	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.83	1.75	1.52	1.00
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.05	1.94	1.64	1.00
	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.32	2.18	1.79	1.00

F – Distribution – 5%, $2\frac{1}{2}$% and 1% points

The table gives, for the F-distribution having ν_1 degrees of freedom in the numerator and ν_2 degrees of freedom in the denominator, the values of F which have respectively a 5%, $2\frac{1}{2}$% and 1% chance of being exceeded. Thus, for the F–distribution with 5 degrees of freedom in the numerator and 9 degrees of freedom in the denominator, there is a 5% chance of $F > 3.48$, a $2\frac{1}{2}$% chance of $F > 4.48$ and a 1% chance of $F > 6.06$.